2969. [2004 : 368, 371] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, c, d, and r be positive real numbers such that $r = \sqrt[4]{abcd} \geq 1$. Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \;\geq\; \frac{4}{(1+r)^2}\,.$$

Solution by Arkady Alt, San Jose, CA, USA.

I suggest the following generalization. For any natural $n \geq 2$, let $a_1, a_2, \ldots, a_n > 0$ such that $a_1 a_2 \cdots a_n = r^n$. Then

$$\frac{1}{(1+a_1)^2} + \frac{1}{(1+a_2)^2} + \cdots + \frac{1}{(1+a_n)^2} \;\geq\; \frac{n}{(1+\sqrt[n]{a_1 a_2 \cdots a_n})^2}$$

if and only if $r \geq \sqrt{n} - 1$.

[*Ed.:* In fact, the condition is not sufficient when $n = 2$. It is possible to find $\varepsilon > 0$ such that $a_1 = \sqrt{2} - 1$, $a_2 = a_1 + \varepsilon$, and $r > \sqrt{2} - 1$, but the inequality fails. The slightly stronger condition $r \geq 0.5$ is sufficient when $n = 2$. Moreover, the inductive step still holds for $n = 2$ using this stronger condition. That is, for $n > 2$, $r \geq \sqrt{n} - 1$ is sufficient for the inequality to hold. The editor has not determined the minimum sufficient value of r in the case $n = 2$.]

We begin with necessity. From the supposition that the inequality holds for all $a_1, a_2, \ldots, a_n > 0$ with $a_1 a_2 \cdots a_n = r^n$, and by setting $a_1 = a_2 = \cdots = a_{n-1} = m$, $a_n = \dfrac{r^n}{m^{n-1}}$, for $m \in \mathbb{R}^+$, we obtain

$$\frac{n-1}{(1+m)^2} + \frac{m^{2(n-1)}}{(m^{n-1}+r^n)^2} \;\geq\; \frac{n}{(1+r)^2}\,,$$

which holds for all positive m. Thus,

$$\lim_{m\to\infty}\left(\frac{n-1}{(1+m)^2} + \frac{m^{2(n-1)}}{(m^{n-1}+r^n)^2}\right) \;=\; 1 \;\geq\; \frac{n}{(1+r)^2}\,,$$

which implies $r \geq \sqrt{n} - 1$.

We prove sufficiency by mathematical induction on $n \geq 2$.

Let $n = 2$ and $a, b > 0$ such that $ab = r^2$ with $r \geq 0.5$.

Set $x = a + b$. Then $x \geq 2r$. Since

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} \;=\; \frac{2 + 2(a+b) + (a+b)^2 - 2ab}{(1+a+b+ab)^2}\,,$$

the inequality can rewritten in the form:

$$\frac{2 + 2x + x^2 - 2r^2}{\left(1 + r^2 + x\right)^2} \;\geq\; \frac{2}{(1+r)^2}$$

$$(1+r)^2\left(2 + 2x + x^2 - 2r^2\right) \;\geq\; 2\left(1 + x + r^2\right)^2\,.$$

This inequality holds if and only if

$$\begin{aligned} 0 &\leq (1+r)^2\left(2+2x+x^2-2r^2\right)-2\left(1+x+r^2\right)^2 \\ &= x^2\left((1+r)^2-2\right)-2x\left(2\left(1+r^2\right)-(1+r)^2\right) \\ &\qquad + \left(2-2r^2\right)(1+r)^2-2\left(1+r^2\right)^2 \\ &= x^2\left(r^2+2r-1\right)-2x\left(r^2-2r+1\right)-4r^4-4r^3-4r^2+4r \\ &= (x-2r)\left(x\left(r^2+2r-1\right)+2\left(r^3+r^2+r-1\right)\right) . \end{aligned}$$

Since $r^2+2r-1 \geq 0$ (this follows from $r \geq \sqrt{2}-1$) and $x \geq 2r$, we have

$$\begin{aligned} x\left(r^2+2r-1\right)+2\left(r^3+r^2+r-1\right) & \\ &\geq 2r\left(r^2+2r-1\right)+2r^3+2r^2+2r-2 \\ &= 2\left(2r^3+3r^2-1\right) = 2(r+1)^2(2r-1) \geq 0 . \end{aligned}$$

Thus,

$$(x-2r)\left(x\left(r^2+2r-1\right)+2\left(r^3+r^2+r-1\right)\right) \geq 0 .$$

Let $a_1, a_2, \ldots, a_n, a_{n+1} > 0$ and $a_1a_2\cdots a_{n+1} = r^{n+1}$, where $r \geq \sqrt{n+1}-1$. Due to symmetry of the inequality, we can suppose that $a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} > 0$.

Set $x = \sqrt[n]{a_1a_2\cdots a_n}$; then $a_{n+1} = \dfrac{r^{n+1}}{x^n}$. Since

$$x \geq a_{n+1} \qquad \Longleftrightarrow \qquad x^{n+1} \geq r^{n+1} \qquad \Longleftrightarrow \qquad x \geq r ,$$

we have $x \geq r$.

Given $x \geq \sqrt{n+1}-1 > \sqrt{n}-1$ and the induction hypothesis, we obtain the inequality:

$$\frac{1}{(1+a_1)^2}+\frac{1}{(1+a_2)^2}+\cdots+\frac{1}{(1+a_n)^2} \geq \frac{n}{(1+x)^2} .$$

[*Ed.:* Note that, for $n=2$, we have $x \geq \sqrt{3}-1 > 0.5$; hence, the inequality does indeed hold. For $n > 2$, $x > \sqrt{n}-1$.]

Therefore,

$$\begin{aligned} \frac{1}{(1+a_1)^2}+\frac{1}{(1+a_2)^2}+\cdots+\frac{1}{(1+a_n)^2}+\frac{1}{(1+a_{n+1})^2} & \\ \geq \frac{n}{(1+x)^2}+\frac{x^{2n}}{\left(x^n+r^{n+1}\right)^2} , \end{aligned}$$

and it is enough to prove that, for all $x \geq r \geq \sqrt{n+1}-1$,

$$\frac{n}{(1+x)^2}+\frac{x^{2n}}{\left(x^n+r^{n+1}\right)^2} \geq \frac{n+1}{(1+r)^2} .$$

Let $h(x) = \frac{n}{(1+x)^2} + \frac{x^{2n}}{(x^n+r^{n+1})^2}$. Then

$$h'(x) \;=\; \frac{2n\left(x^{n+1}-r^{n+1}\right)\left(x^{n+1}r^{n+1}+3x^nr^{n+1}+r^{2n+2}-x^{2n-1}\right)}{(1+x)^3\left(x^n+r^{n+1}\right)^3}\,.$$

Now everything depends on the behaviour of the polynomial

$$P_n(x) \;=\; x^{n+1}r^{n+1}+3x^nr^{n+1}+r^{2n+2}-x^{2n-1}\,.$$

Note that

$$\begin{aligned} x^{n+1}r^{n+1}+3x^nr^{n+1}+r^{2n+2}-x^{2n-1} &\;=\; 0 \\ \text{or} \qquad r^{n+1}+\frac{3r^{n+1}}{x}+\frac{r^{2n+2}}{x^{n+1}}-x^{n-2} &\;=\; 0\,. \end{aligned}$$

Set $\phi(x) = r^{n+1}+\frac{3r^{n+1}}{x}+\frac{r^{2n+2}}{x^{n+1}}-x^{n-2}$.
Since $r \ge \sqrt{n+1}-1 > \frac{1}{2}$ for $n \ge 2$, we have

$$\begin{aligned} P_n(r) &\;=\; 2r^{2n+2}+3r^{2n+1}-r^{2n-1} \\ &\;=\; r^{2n-1}\left(2r^3+3r^2-1\right) \\ &\;=\; r^{2n-1}(r+1)^2(2r-1) \;>\; 0 \\ \iff \qquad \phi(r) &\;>\; 0\,. \end{aligned}$$

Since $\phi(x)$ is continuous on $(0,\infty)$, $\phi(x)$ strictly decreases on $[r,\infty)$, and $\phi(\infty)\phi(r) < 0$, there is only one point, x_0, in (r,∞) such that $\phi(x_0) = 0$, or equivalently $P_n(x_0) = 0$.

Moreover, $\phi(x) > \phi(x_0) = 0$ is equivalent to $P_n(x) > 0$ for all $x \in [r,x_0)$, and $0 = \phi(x_0) > \phi(x)$ is equivalent to $P_n(x) < 0$ for all $x \in (x_0,\infty)$.

Since

$$\min_{x\in[r,x_0]} h(x) \;=\; h(r) \;=\; \frac{n}{(1+r)^2}+\frac{r^{2n}}{\left(r^n+r^{n+1}\right)^2} \;=\; \frac{n+1}{(1+r)^2}\,,$$

and, for any $x \in [x_0,\infty)$,

$$h(x) \;>\; \lim_{x\to\infty} h(x) \;=\; 1 \;\ge\; \frac{n+1}{(1+r)^2} \;=\; h(r)\,,$$

we obtain

$$\min_{x\in[r,\infty)} h(x) \;=\; h(r) \;=\; \frac{n+1}{(1+r)^2}\,.$$

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